

# Properties of proper rational numbers

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October 3, 2011

## 1 Introduction

The set of rational numbers can be thought of as the disjoint union of two of its main subsets: the set of integers and the set of proper rationals.

**Definition 1:** A proper rational number is a rational number which is not an integer.

The aim of this work is simple and direct. Namely, to explore some of the basic or elementary properties of the proper rationals.

We will make use of the standard notation  $(u, w)$  denoting the greatest common divisor of two integers  $u$  and  $w$ . Also, the notation  $u|w$  to denote that  $u$  is a divisor of  $w$ .

**Proposition 1.** *Let  $r$  be a proper rational number. Then  $r$  can be written in the form,  $r = \frac{c}{b}$  where  $c$  and  $b$  are relatively prime integers;  $(c, b) = 1$ , and with  $b \geq 2$ .*

*Proof.* Since  $r$  is a proper rational, it cannot, by definition, be zero. Hence  $r = \frac{A}{B}$ , for some positive integers  $A$  and  $B$ ; if  $r > 0$ . If, on the other hand,  $r < 0$ , then  $r = -\frac{A}{B}$ ,  $A, B$  being positive integers. Let  $d = (A, B)$ , then  $A = da$ ,  $B = db$ , for relative prime positive integers  $a$  and  $b$ . We have,  $r = \frac{A}{B} = \frac{da}{db} = \frac{a}{b}$ , for  $r > 0$ . Clearly,  $b$  cannot equal 1, for then  $r$  would equal  $a$ , an integer, contrary to the fact that  $r$  is a proper rational. Hence,  $b \geq 2$ .

If, on the other hand,  $r < 0$ ,  $r = \frac{-A}{B} = \frac{-a}{b} = \frac{c}{b}$  with  $c = -a$ , and  $b \geq 2$ .  $\square$

**Definition 2:** A proper rational number  $r$  is said to be in standard form if it is written as  $r = \frac{c}{b}$ , where  $c$  and  $b$  are relatively prime integers and  $b \geq 2$

## 2 The reciprocal of a proper rational

We state the following result without proof. We invite the interested reader to fill in the details.

**Theorem 1.** *Let  $r = \frac{c}{b}$  be a proper rational in standard form.*

- (i) *If  $|c| = 1$ , the reciprocal  $\frac{1}{r} = \frac{b}{c}$  is an integer.*
- (ii) *If  $c \geq 2$ , the reciprocal  $\frac{1}{r}$  is a proper positive rational.*
- (iii) *If  $|c| \geq 2$  and  $c < 0$ , the reciprocal  $\frac{1}{r}$  is negative proper rational with the standard form being  $\frac{1}{r} = \frac{d}{|c|}$ , where  $d = -b$ .*

## 3 An obvious property

Is the sum of a proper rational and an integer always a proper rational? The answer is a rather obvious yes.

**Theorem 2.** *Suppose that  $r = \frac{c}{b}$  is a proper rational in standard form; and  $d$  an integer. Then the sum  $r + d$  is a proper rational.*

*Proof.* If, to the contrary,  $r + d = i$ , for some  $i \in \mathbb{Z}$ , then  $r = i - d$ , an integer contradicting the fact that  $r$  is a proper rational.  $\square$

## 4 A lemma from number theory

We will make repeated use of the very well known, and important, lemma below. For a proof of this lemma, see reference [1]. It can be found in just about every elementary number theory book.

**Lemma 1.** (*Euclid's lemma*)

- (i) (*Standard version*) Let  $m, n, k$  be positive integers such that  $m$  is a divisor of the product  $n \cdot k$ ; and suppose that  $(m, n) = 1$ . Then  $m$  is a divisor of  $k$ .
- (ii) (*Extended version*) Let  $m, n, k$  be non-zero integers such that  $m|nk$  and  $(m, n) = 1$ . Then  $m|k$ .

## 5 A slightly less obvious property

When is the product of a proper rational with an integer, an integer? A proper rational?

**Theorem 3.** Let  $r = \frac{c}{b}$  be a proper rational in standard form and  $i$  an integer.

- (a) The product  $r \cdot i$  is an integer if, and only if,  $b|i$ .
- (b) The product  $r \cdot i$  is a proper rational if, and only if,  $b$  is not a divisor of  $i$ .

*Proof.* (b) This part is logically equivalent to part (a).

(a) If  $b$  divides  $i$ , then  $i = b \cdot q$ , an integer.

so we have  $r \cdot i = \frac{cbq}{b} = c \cdot q$ , an integer. Now the converse. Suppose that  $r \cdot i$  is an integer  $t$ :  $r \cdot i = t$ , which yields,

$$c \cdot i = b \cdot t \tag{1}$$

Since  $r$  is a proper rational,  $(b, c) = 1$  by definition. Equation (1) shows that  $b|c \cdot i$ ; and since  $(b, c) = 1$ . Lemma 1 implies that  $b$  must divide  $i$ . We are done.  $\square$

## 6 The sum of two proper rationals

An interesting equation arises. When is the sum of two proper rationals also a proper rational? When is it an integer? There is no obvious answer here.

**Theorem 4.** *Let  $r_1 = \frac{c_1}{b_1}$  and  $r_2 = \frac{c_2}{b_2}$  be proper rationals in standard form. Then,*

- (i) *The sum  $r_1 + r_2$  is an integer if, and only if,  $b_1 = b_2$  and  $b_1$  is a divisor of the sum  $c_1 + c_2$ .*
- (ii) *The sum  $r_1 + r_2$  is a proper rational if, and only if, either  $b_1 \neq b_2$  or  $b_1 = b_2$  but with  $b_1$  not being a divisor of  $c_1 + c_2$ .*

*Proof.* (ii) This part is logically equivalent to part (i).

(i) If  $b_1 = b_2$  and  $b_1 | (c_1 + c_2)$ , then  $r_1 + r_2 = \frac{c_1 + c_2}{b_1}$ , is obviously an integer. Next, let us prove the converse statement.

Suppose that  $r_1 + r_2 = i$ , an integer. Some routine algebra produces

$$c_1 b_2 + c_2 b_1 = i b_1 b_2 \quad (2)$$

or equivalently

$$c_1 b_2 = b_1 (i b_2 - c_2). \quad (3)$$

According to (3),  $b_1 | c_1 b_2$ ; and since  $(b_1, c_1) = 1$ , Lemma 1 implies that  $b_1 | b_2$ . A similar argument, using equation (2), once more establishes that  $b_2 | b_1$ . Clearly, since the two positive integers  $b_1$  and  $b_2$  are divisors of each other, they must be equal;  $b_1 = b_2$  (an easy exercise in elementary number theory). From  $b_1 = b_2$  and (2), we obtain  $c_1 + c_2 = i \cdot b_i$ ; and thus it is clear that  $b_1 | (c_1 + c_2)$ .  $\square$

## 7 The product of two proper rationals

**Theorem 5.** *Let  $r_1 = \frac{c_1}{b_1}$  and  $r_2 = \frac{c_2}{b_2}$  be proper rationals in standard form.*

- (a) *The product  $r_1 r_2$  is an integer if, and only if,  $b_1 | c_2$  and  $b_2 | c_1$ .*
- (b) *The product  $r_1 r_2$  is a proper rational if, and only if,  $b_1$  is not a divisor of  $c_2$ ; or  $b_2$  is not a divisor of  $c_1$ .*

*Proof.* (b) This part is logically equivalent to part (a).

(a) Suppose that  $b_1|c_2$  and  $b_2|c_1$ ; then  $c_2 = b_1a$  and  $c_1 = b_2d$  where  $a$  and  $d$  are (non-zero) integers.

We have  $r_1r_2 = \frac{c_1c_2}{b_1b_2} = \frac{adb_1b_2}{b_1b_2} = ad$ , an integer.

Conversely, suppose that  $r_1r_2 = i$ , an integer. Then

$$c_1c_2 = ib_1b_2 \quad (4)$$

Since  $(b_1, c_1) = 1 = (c_2, b_2)$ , (4), in conjunction with Lemma 1, imply that  $b_1|c_2$  and  $b_2|c_1$ . We are done.  $\square$

## 8 One more result and its corollary

In Theorem 4 part (i), gives us the precise conditions for the sum of two proper rationals to be an integer. Likewise, Theorem 5 part (a) gives us the exact conditions for the product of two proper rationals to be an integer. Naturally, the following question arises. Can we find two proper rational numbers whose sum is an integer; and also whose product is an integer? Theorem 7 provides an answer in the negative. Theorem 7 is a direct consequence of Theorem 6 below.

**Theorem 6.** *If both the sum and the product of two rational numbers are integers, then so are the two rationals, integers.*

*Proof.* Let  $r_1, r_2$  be the two rationals, and suppose that

$$\left\{ \begin{array}{rcl} r_1 + r_2 & = & i_1 \\ r_1r_2 & = & i_2 \\ i_1, i_2 \in \mathbb{Z} \end{array} \right\} \quad (5)$$

If either of  $r_1, r_2$  is an integer, then the first equation in (5) implies that the other one is also an integer. So we are done in this case. So, assume that neither of  $r_1, r_2$  is an integer; which means that they are both proper rationals. Let then  $r_1 = \frac{c_1}{b_1}$ ,  $r_2 = \frac{c_2}{b_2}$  be the standard forms of  $r_1$  and  $r_2$ . That is,  $(c_1, b_1) = 1 = (c_2, b_2)$ ,  $b_1 \geq 2$ ,  $b_2 \geq 2$  and, of course,  $c_1c_2 \neq 0$ .

Combining this information with (5), we get

$$\left\{ \begin{array}{rcl} c_1b_2 + c_2b_1 & = & i_1b_1b_2 \\ c_1c_2 & = & i_2b_1b_2 \end{array} \right\} \quad (6)$$

From the first equation in (6) we obtain

$$c_1 b_2 = b_1 (i_1 b_2 - c_2),$$

which shows that  $b_1 | c_1 b_2$ .

This, combined with  $(c_1, b_1) = 1$  and Lemma 1 allow us to deduce that  $b_1 | b_2$ . Similarly, using the first equation in (6), we infer that  $b_2 | b_1$  which implies  $b_1 = b_2$ . Hence, the second equation of (6) gives,

$$c_1 c_2 = i_2 b_1^2 \tag{7}$$

By virtue of  $(b_1, c_1) = (b_1, c_2) = 1$ , equation (7) implies  $b_1 = 1$ ;  $b_1 = b_2 = 1$ . Therefore  $r_1$  and  $r_2$  are integers.  $\square$

We have the immediate corollary.

**Theorem 7.** *There exist no two proper rationals both of whose sum and product are integers.*

## 9 A closing remark

Theorem 7 can also be proved by using the well known Rational Root Theorem for polynomials with integer coefficients. The Rational Root Theorem implies that if a monic (i.e., leading coefficient is 1) polynomial with integer coefficients has a rational root that root must be an integer. Every rational of such a monic polynomial must be an integer (equivalently, each of its real roots, if any, must be either an irrational number or an integer). Thus, in our case, the rational numbers  $r_1$  and  $r_2$  are the roots of the monic trinomial,  $t(x) = (x - r_1)(x - r_2) = x^2 - i_1 x + i_2$ ; a monic quadratic polynomial with integer coefficients  $-i_1$  and  $i_2$ . Hence,  $r_1$  and  $r_2$  must be integers.

For more details, see reference [1].

## References

- [1] 1 Kenneth H. Rose, *Elementary Numbers Theory and Its Applications*, fifth edition, 2005, Pearson-Addison-Wesley.

For Lemma 1 (Lemma 3.4 in the above book), see page 109 for the Rational Root Theorem, see page 115.